Int. J. Solids Structures Vol. 15, pp. 325–334

0020-7683/79/0401-0325/$02.00/0

MIXTURE THEORY FOR LONGITUDINAL WAVE PROPAGATION IN UNIDIRECTIONAL COMPOSITES WITH CYLINDRICAL FIBERS OF ARBITRARY CROSS SECTION—I

FORMULATION†

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(*Received* 27 *March*, 1978; *in revised form* 18 *September* 1978; *received for publication* 31 *October* 1978)

Abstract—A binary mixture theory is developed for propagation of longitudinal waves in a unidirectional, fibrous composite containing a two-dimensional periodic array of cylindrical fibers of *arbitrary* cross section. The case considered concerns a class of problems for which the waves propagate in the direction of the fiber axis. Model construction is based upon an asymptotic scheme wherein the signal wavelength is assumed to be large compared to composite micro-dimension; this premise is appropriate for many applications. The resulting theory, which retains information on the displacement profile in the microstructure, is closed in the sense that all the mixture properties that it contains can be determined once the geometry of the unit cell, shape of the fiber cross section and the properties of the constituents are known.

In the theory formulated herein, calculations of the mixture properties require the solution of time-independent microstructure boundary value problems (MBVPs) defined over a unit cell. In this part of the paper, the MBVPs are derived, and the relationship between their solution and mixture properties is established. The description of a finite-element procedure to solve the MBVPs is deferred to the second part of the paper, which also contains an application of mixture theory to a study of dispersion of time-harmonic waves in composites with several microstructural geometries and combination of material properties.

INTRODUCTION

In this two-part[1] paper, a binary mixture theory is developed for propagation of longitudinal waves in a unidirectional composite reinforced by a general two-dimensional periodic array of cylindrical fibers of *arbitrary* cross section. The case considered concerns a class of problems for which the waves propagate in the direction of the fiber axes. The development of the theory is based upon a technique introduced by Hegemier[2], and generalized recently by the authors[3] in the context of a similar heat-conduction problem. The main advantage of the technique is that it takes into account the heterogeneous character and the microstructure of the material, including the shape of the fiber cross section, in a systematic manner. The importance given here to the ability of our theory to include the effect of the fiber geometry on transient phenomena in fibrous composites arises not only from a theoretical interest in such effects but also from the simple consideration that, in applications, fibers are almost never perfectly circular. Indeed, it is characteristic of some modern manufacturing techniques used in industry that they yield nearly rectangular fibers in the final product.

Mixture theories have been discussed in a more general framework by Green and Naghdi[4] and Bowen[5], among others, and applied to periodic composities by Bedford and Stern[6]. These general formulations[4, 5], however, usually require that a series of experiments be performed on the composite to determine the constants appearing in the mixture constitutive relations. In contrast, in the method utilized here the constants in the constitutive equations are calculated from the solutions of well-posed, time-independent microstructure boundary value problems defined over a unit cell. The solutions of the microstructure boundary value problems can also be used to calculate the local distribution of the field quantities once the mixture equations have been solved. This procedure, which has been used successfully for a number of wave propagation and thermal diffusion problems[7–9], results in the resolution of the micro-

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structure to a degree of accuracy not achieved by some other theories in which an *a priori* assumption has to be made regarding the distribution of the field variables in the unit cell[10–14] and the relationship between micro- and macro-quantities[14].

Although, in general, the mixture equations that govern the evolution of average displacements have to be solved numerically, their solution is much more economical than direct numerical solution of the original problem. Clearly, this is due to the fact that mixture equations derived in the following development contain one spatial dimension only, as opposed to three in the original problem. This reduction in the number of independent variables can reduce the computing time by more than two orders of magnitude[7].

The outline of the paper is as follows. In the first part, mixture equations are derived for longitudinal wave propagation in composites with cylindrical fibers in general two-dimensional arrays. Using the asymptotic technique introduced by Hegemier[2], the constituent partial stresses and the interaction term that occur in the mixture equations are related to the constituent average displacements. These relationships contain constants which can be calculated from the solution of static microstructure boundary value problems derived herein.

The second part of the paper contains a description of a variational-principle based finite element method for solution of microstructure boundary value problems and detailed parametric studies for a variety of fiber cross sections and unit cell geometries. Based upon the mixture properties calculated in these studies, mixture theory is applied to a study of dispersion of time harmonic waves.

FORMULATION

Basic relations

Consider a periodic, two-dimensional array of unidirectional, cylindrical fibers of arbitrary cross section embedded in a matrix, as illustrated in Fig. 1(a). A "cell" is associated with each fiber as depicted in Fig. 1(b). Each such cell consists of domains $\bar{A}^{(1)}$ and $\bar{A}^{(2)}$ occupied by the fiber and the matrix, respectively. Interface between the fiber and the matrix is denoted by $\bar{\mathscr{I}}$, and $\bar{\mathscr{C}}$ denotes the boundary of the cell.

With respect to the rectangular Cartesian coordinates $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$ shown in Fig. 1(a), the composite is assumed to occupy the domain $0 \le \bar{x}_3 \le \bar{l}$, $-\infty \le \bar{x}_1, \bar{x}_2 \le \infty$. The two constituents of each cell are assumed to be homogeneous and isotropic, bonded perfectly at the interface so that no slip is permitted. In the following analysis longitudinal waves propagating in the fiber axis direction are considered, and the initial and boundary data are assumed to be such that motion is similar in each cell. In view of the last premise only a typical cell needs to be analyzed for construction of an approximate model.

Notations of the form $(\,)^{(\alpha)}$ denote quantities associated with material α, with $\alpha = 1$ or 2, where "material 1" refers to the fiber and "material 2" refers to the matrix. Lower case Latin indices range from 1 to 3, with the usual summation convention, unless noted otherwise. For derivatives, the notations $(\,)_{,i}$ for $\partial(\,)/\partial\bar{x}_i$ and $\dot{(\,)}$ for $\partial(\,)/\partial\bar{t}$ are used, with $\bar{t}$ denoting the time.

With the foregoing assumptions and notational convention, the displacement fields $\bar{u}_i^{(\alpha)}$ and stress fields $\bar{\sigma}_{ij}^{(\alpha)}$ in the two constituents can be obtained by the solution of following equations:

(a) *Conservation of momentum*

$$\bar{\sigma}_{ij,i}^{(\alpha)} = \bar{\rho}^{(\alpha)}\ddot{\bar{u}}_j^{(\alpha)}, \qquad \bar{\sigma}_{ij}^{(\alpha)} = \bar{\sigma}_{ji}^{(\alpha)}. \tag{1}$$

(b) *Constitutive relations*

$$\bar{\sigma}_{ij}^{(\alpha)} = \bar{\lambda}^{(\alpha)}\bar{u}_{l,l}^{(\alpha)}\delta_{ij} + \bar{\mu}^{(\alpha)}(\bar{u}_{i,j}^{(\alpha)} + \bar{u}_{j,i}^{(\alpha)}). \tag{2}$$

(c) *Similarity conditions*

$$\bar{u}_i^{(2)}\bar{\nu}_i^{(2)} = 0 \text{ on } \bar{\mathscr{C}}, \tag{3a}$$

$$\epsilon_{ijk}\bar{\sigma}_{lj}^{(2)}\bar{\nu}_l^{(2)}\bar{\nu}_k^{(2)} = 0 \text{ on } \bar{\mathscr{C}}. \tag{3b}$$

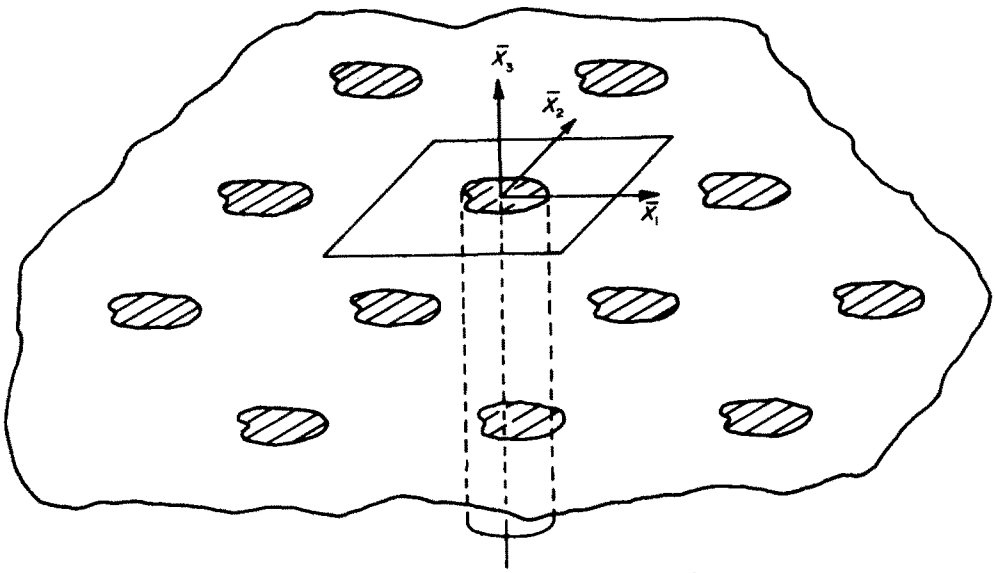

Fig. 1(a). Geometry of fiber-reinforced composite and coordinate system.

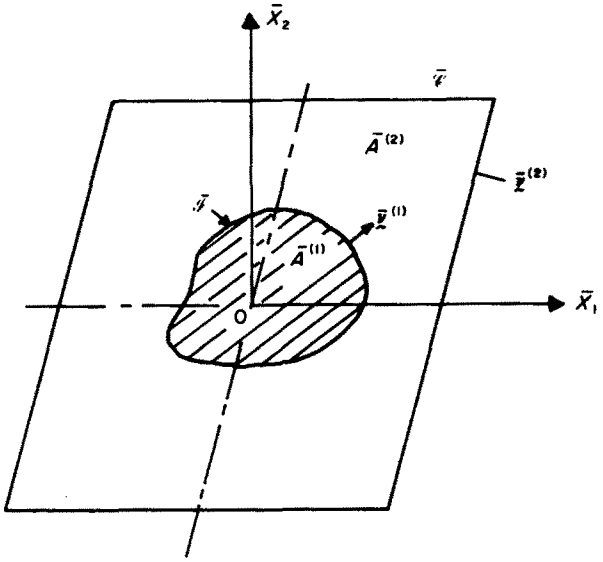

Fig. 1(b). A typical cell geometry.

(d) *Interface continuity conditions*

$$\bar{u}_i^{(1)} = \bar{u}_i^{(2)} \text{ on } \bar{\mathscr{F}}, \tag{4a}$$

$$\bar{\sigma}_{ij}^{(1)}\bar{\nu}_i^{(1)} = \bar{\sigma}_{ij}^{(2)}\bar{\nu}_i^{(1)} \text{ on } \bar{\mathscr{F}}. \tag{4b}$$

In the above set of equations $\bar{\lambda}^{(\alpha)}$, $\bar{\mu}^{(\alpha)}$ are Lame's constants, and $\bar{\nu}_i^{(\alpha)}$ denotes the unit outward normal on a boundary of $\bar{A}^{(\alpha)}$. Since cylindrical fibers are considered, $\bar{\nu}_3^{(\alpha)} = 0$. For complete specification of the boundary value problem that governs the motion of the constituents, one must also specify

(e) *Initial conditions at* $\bar{t} = 0$ *and appropriate boundary data at* $\bar{x}_3 = 0, \bar{l}$; *where* $\bar{x}_3 \in [0, \bar{l}]$.

Obviously, the conditions (a)–(e) specify a well posed time dependent boundary value problem in a three dimensional domain. The object of the following development is to derive the differential equations governing macroscopic wave propagation in a systematic manner such that they involve one spatial dimension only and yet reflect, to a desired degree of accuracy, the effect of momentum transfer in the micro-scale. For this purpose a mixture theory is formulated

which is based upon an asymptotic scheme under the assumption that the characteristic length $\bar{\Delta}$ of a cell is small compared to a macrodimension $\bar{\Lambda}$ which is $0(\bar{l})$ if $\bar{l}$ is finite. The analysis is facilitated by a judicious scaling of various dependent and independent variables. To do so, first the following notation is introduced: $\bar{\Lambda}$: typical macrodimension; $\bar{\Delta}$: typical composite cell length; $\bar{c}_{(m)}$, $\bar{\rho}_{(m)}$: reference wave speed and mass density of the composite; $\bar{E}_{(m)} = \bar{\rho}_{(m)}\bar{c}^2_{(m)}$: reference elastic modulus of the composite; $\bar{t}_0 = \bar{\Lambda}/\bar{c}_{(m)}$: typical signal travel time in the axial direction; and $\epsilon = \bar{\Delta}/\bar{\Lambda}$: ratio of micro-to-macro dimensions, where $\bar{\rho}_{(m)}$ and $\bar{E}_{(m)}$ are mixture properties to be defined subsequently. The reference scales $\bar{t}_0$ and $\bar{\Lambda}$ are also related to "observation" time scale and length scale in the following sense. Upon application of a stress or displacement pulse, the mixture theory shall be able to yield reliable results only after a time interval which is not too small in comparison to $\bar{t}_0$. In a similar manner, the mixture theory results are not expected to be very accurate at points in the composite whose distances from the boundaries are too small compared to $\bar{\Lambda}$. Thus, for the purpose of scaling, instead of the use of an actual axial dimension or a dimension associated with a time scale characteristic of input pulse, it is also possible to utilize an observation length or time scale which reflects the fineness to which the resolution of deformation history is sought.

With the notation just introduced, the following nondimensional quantities are defined:

$$(\epsilon x_i, x_3) = (\bar{x}_i, \bar{x}_3)/\bar{\Lambda}, \qquad i = 1, 2, \tag{5a}$$

$$\rho^{(\alpha)} = \bar{\rho}^{(\alpha)}/\bar{\rho}_{(m)}, \qquad (\lambda^{(\alpha)}, \mu^{(\alpha)}) = (\bar{\lambda}^{(\alpha)}, \bar{\mu}^{(\alpha)})/\bar{E}_{(m)}, \tag{5b}$$

$$(\sigma^{(\alpha)}_{ij}, \epsilon\sigma^{(\alpha)}_{3i}, \sigma^{(\alpha)}_{12}) = (\bar{\sigma}^{(\alpha)}_{ij}, \bar{\sigma}^{(\alpha)}_{3i}, \bar{\sigma}^{(\alpha)}_{12})/\bar{E}_{(m)}, \quad j = 1, 3;\ i = 1, 2, \text{ no sum on } j, \tag{5c}$$

$$(\epsilon u_i^{(\alpha)}, u_3^{(\alpha)}) = (\bar{u}_i^{(\alpha)}, \bar{u}_3^{(\alpha)})/\bar{\Lambda}, \qquad i = 1, 2. \tag{5d}$$

The basic equations (1)–(4) for the scaled variables defined in (5) can be rewritten as follows:

(a) *Conservation of momentum*

$$\sigma^{(\alpha)}_{j3,j} = \rho^{(\alpha)}\ddot{u}_3^{(\alpha)}, \tag{6a}$$

$$\epsilon^2\sigma^{(\alpha)}_{3i,3} + \sigma^{(\alpha)}_{ki,k} = \epsilon^2\rho^{(\alpha)}\ddot{u}_i^{(\alpha)}, \qquad i, k = 1, 2, \tag{6b}$$

$$\sigma^{(\alpha)}_{ij} = \sigma^{(\alpha)}_{ji}. \tag{7}$$

(b) *Constitutive relations*

$$\sigma^{(\alpha)}_{ij} = \lambda^{(\alpha)}u^{(\alpha)}_{i,i} + 2\mu^{(\alpha)}u^{(\alpha)}_{j,j}, \qquad i, j = 1, 2, 3, \text{ no sum on } j, \tag{8a}$$

$$\epsilon^2\sigma^{(\alpha)}_{i3} = \mu^{(\alpha)}(\epsilon^2 u^{(\alpha)}_{i,3} + u^{(\alpha)}_{3,i}), \qquad i = 1, 2, \tag{8b}$$

$$\sigma^{(\alpha)}_{12} = \mu^{(\alpha)}(u^{(\alpha)}_{1,2} + u^{(\alpha)}_{2,1}) \tag{8c}$$

(c) *Similarity conditions*

$$u_i^{(2)}\nu_i^{(2)} = 0 \text{ on } \mathscr{C}, \tag{9a}$$

$$\sigma^{(2)}_{i3}\nu_i^{(2)} = 0, \quad \epsilon_{3ik}\sigma^{(2)}_{li}\nu_l^{(2)}\nu_k^{(2)} = 0 \text{ on } \mathscr{C}. \tag{9b}$$

(d) *Interface conditions*

$$u_3^{(1)} = u_3^{(2)}, \epsilon u_i^{(1)} = \epsilon u_i^{(2)} \text{ on } \mathscr{S}, \qquad i = 1, 2, \tag{10a}$$

$$\epsilon\sigma^{(1)}_{i3}\nu_i^{(1)} = \epsilon\sigma^{(2)}_{i3}\nu_i^{(1)}, \sigma^{(1)}_{ik}\nu_i^{(1)} = \sigma^{(2)}_{ik}\nu_i^{(1)} \text{ on } \mathscr{S}, \qquad k = 1, 2. \tag{10b}$$

(e) ***Initial and boundary conditions.***

In the above $A^{(\alpha)}$, $\mathscr{S}$, $\mathscr{C}$, $\nu_i^{(\alpha)}$ denote $\bar{A}^{(\alpha)}$, $\bar{\mathscr{S}}$, $\bar{\mathscr{C}}$, $\bar{\nu}_i^{(\alpha)}$, respectively, in nondimensional coordinates; partial derivatives are now defined by $(\ )_{,i} \equiv \partial(\ )/\partial x_i$, $(\dot{})\equiv \partial(\ )/\partial t$.

Mixture equations

To obtain the mixture equations, averaged quantities are defined as

$$f^{(\alpha a)} = \frac{1}{A^{(\alpha)}} \iint_{A^{(\alpha)}} f^{(\alpha)}(x_1, x_2, x_3, t)\, dx_1\, dx_2. \tag{11}$$

The momentum equation (6a) in x_3 direction is now averaged according to (11), so that with the use of the Gauss Theorem and similarity conditions

$$(A^{(\alpha)}\sigma_{33}^{(\alpha a)})_{,3} + \oint_{\mathscr{S}} \sigma_{i3}^{(\alpha)} \nu_i^{(\alpha)}\, ds = \rho^{(\alpha)} A^{(\alpha)} \ddot{u}_3^{(\alpha a)}. \tag{12}$$

Equation (12) suggests the definitions of "partial" stress and "partial" density according to

$$\sigma_{33}^{(\alpha p)} = n^{(\alpha)} \sigma_{33}^{(\alpha a)}, \tag{13a}$$

$$\rho^{(\alpha p)} = n^{(\alpha)} \rho^{(\alpha)} \tag{13b}$$

where $n^{(\alpha)}$ is the volume fraction of the α constituent, given by

$$n^{(\alpha)} = A^{(\alpha)}/(A^{(1)} + A^{(2)}) = \bar{A}^{(\alpha)}/(\bar{A}^{(1)} + \bar{A}^{(2)}) = A^{(\alpha)}/A, \tag{14}$$

with A denoting the area of the cell in the scaled system. If (13) is used in (12), together with the fact that $\nu_i^{(2)} = -\nu_i^{(1)}$ on $\mathscr{S}$, the following familiar binary mixture form is obtained

$$\sigma_{33,3}^{(1p)} - \rho^{(1p)} \ddot{u}_3^{(1a)} = -P, \tag{15a}$$

$$\sigma_{33,3}^{(2p)} - \rho^{(2p)} \ddot{u}_3^{(2a)} = P, \tag{15b}$$

where

$$P = \frac{1}{A} \oint_{\mathscr{S}} \sigma_{3i}^{(1)} \nu_i^{(1)}\, ds = \frac{1}{A} \oint_{\mathscr{S}} \sigma_{3i}^{(2)} \nu_i^{(1)}\, ds. \tag{16}$$

Physical interpretation of the so-called interaction term P in (15) is obvious: it reflects the momentum transfer that takes place between the two constituents through the axial shear on the interface.

Having been derived from the exact axial momentum equation, the mixture equations (15) entail no approximation at this stage. However, in order to obtain a closed theory it is also necessary to determine the functional dependence of P and $\sigma_{33}^{(\alpha p)}$ on the averaged constituent displacements. It is in the construction of these mixture constitutive relations that approximations have to be introduced. For this purpose, the method introduced by Hegemier[2] is the most convenient in that it provides a systematic procedure for incorporating the effect of microstructure on the mixture constitutive equations. This method is based on an asymptotic scheme which shall now be pursued.

Asymptotic expansions

The parameter ϵ introduced in a previous section is the ratio of signal travel times in the transverse and axial directions. The fundamental premise is now introduced that this parameter is small compared to unity. This premise, together with the form of the scaled equations, suggests the following regular expansions for the dependent variables:

$$u_i^{(\alpha)}(x_l, t; \epsilon) = \sum_{n=0}^{\infty} \epsilon^{2n} u_{i(2n)}^{(\alpha)}(x_l, t), \tag{17a}$$

$$\sigma_{ij}^{(\alpha)}(x_l, t; \epsilon) = \sum_{n=0}^{\infty} \epsilon^{2n} \sigma_{ij(2n)}^{(\alpha)}(x_l, t). \tag{17b}$$

Implicit in the subsequent use of this expansion is the assumption that all the scaled material properties in (6)–(10) are of order unity. This can be insured for composites with achievable volume fractions through a proper choice of mixture properties used for scaling; this will be accomplished in a later section.

It should be emphasized here that because of the *a priori* assumption on the order of magnitude of various field quantities in (5) and due to the form of the assumed expansions (17), the mixture theory is expected to yield an *outer* solution to the problem under study. As a result, there are layers near the boundaries wherein the mixture theory is not strictly applicable. However, both within and outside these boundary layers, whose thicknesses are of the order of composite microdimension, the accuracy of the mixture theory is better than that of the effective modulus theory which treats the composite as a homogeneous material. This greater degree of accuracy of the mixture theory is of particular significance for wave-propagation problems, since it enables the theory to predict the dominant phenomenon of dispersion, which, obviously, is absent in the effective modulus model.

If (17) are substituted into (6)–(10) and coefficients of different powers of ϵ are set equal to zero, a system of equations for each $n = 0, 1, 2 \ldots$ is obtained. The first order equations consist of two time independent problems over a unit cell in the x_1, x_2 plane. With the stipulation that in all of the subsequent analysis *lower case Latin indices range from* 1 *to* 2, these problems are:

(i) *In-plane stress problem*

$$\sigma^{(\alpha)}_{ij(0),i} = 0, \tag{18}$$

$$\sigma^{(\alpha)}_{ij(0)} = \delta_{ij}\lambda^{(\alpha)}(u^{(\alpha)}_{l(0),l} + u^{(\alpha)}_{3(0),3}) + \mu^{(\alpha)}(u^{(\alpha)}_{i(0),j} + u^{(\alpha)}_{j(0),i}) \tag{19}$$

$$u^{(2)}_{i(0)}\nu^{(2)}_i = 0, \quad \epsilon_{3ij}\sigma^{(2)}_{kl}\nu^{(2)}_k\nu^{(2)}_j = 0 \text{ on } \mathscr{C}, \tag{20}$$

$$u^{(1)}_{i(0)} = u^{(2)}_{i(0)}, \quad \sigma^{(1)}_{ij(0)}\nu^{(1)}_i = \sigma^{(2)}_{ij(0)}\nu^{(1)}_i \text{ on } \mathscr{S}. \tag{21}$$

(ii) *Axial shear problem*

$$\sigma^{(\alpha)}_{i3(0),i} = \rho^{(\alpha)}\ddot{u}^{(\alpha)}_{3(0)} - \sigma^{(\alpha)}_{33(0),3}, \tag{22}$$

$$\sigma^{(\alpha)}_{33(0)} = (\lambda^{(\alpha)} + 2\mu^{(\alpha)})u^{(\alpha)}_{3(0),3} + \lambda^{(\alpha)}u^{(\alpha)}_{i(0),i}, \tag{23}$$

$$\sigma^{(\alpha)}_{3i(0)} = \mu^{(\alpha)}(u^{(\alpha)}_{i(0),3} + u^{(\alpha)}_{3(2),i}), \quad \sigma^{(\alpha)}_{i3(0)} = \sigma^{(\alpha)}_{3i(0)}, \tag{24}$$

$$\sigma^{(2)}_{i3(0)}\nu^{(2)}_i = 0 \text{ on } \mathscr{C}, \tag{25}$$

$$u^{(1)}_{3(0)} + \epsilon^2 u^{(1)}_{3(2)} = u^{(2)}_{3(0)} + \epsilon^2 u^{(2)}_{3(2)} \text{ on } \mathscr{S}, \tag{26}$$

$$\sigma^{(1)}_{i3(0)}\nu^{(1)}_i = \sigma^{(2)}_{i3(0)}\nu^{(1)}_i \text{ on } \mathscr{S}. \tag{27}$$

It might be pointed out here that the satisfaction of the condition of continuity of the axial displacement at the interface to $0(\epsilon^2)$, as in (26), is essential to have enough boundary conditions for $u^{(\alpha)}_{3(2)}$ to be solved. Further, inclusion of the $u^{(\alpha)}_{3(2)}$ terms in theory is imperative if it is to be able to predict dispersive phenomena. A detailed discussion of this point may be found in Hegemier[2].

The zeroth order expansion of (8b) indicates that the lowest order axial displacements are independent of the transverse coordinates, i.e.

$$u^{(\alpha)}_{3(0)}(x_i, x_3, t) = u^{(\alpha)}_{3(0)}(x_3, t). \tag{28}$$

This result considerably simplifies the construction of mixture constitutive relations.

Mixture theory based on lowest order system

(i) *Constitutive relations for partial stresses.* Because of the appearance of the term $u^{(\alpha)}_{3(0),3}$ in (19), the forcing term in the in-plane stress problem (18)–(21) is the jump in the quantity $\lambda^{(\alpha)}u^{(\alpha)}_{3(0),3}$ across $\mathscr{S}$, which maintains the continuity conditions (21b). Since the problem is linear, its solution can be written as a linear function of the jump, denoted by $J(x_3, t)$. Thus, with

$$J(x_3, t) = \lambda^{(2)}u^{(2)}_{3(0),3} - \lambda^{(1)}u^{(1)}_{3(0),3}, \tag{29}$$

new variables $v_i^{(\alpha)}$ and $\tau_{ij}^{(\alpha)}$ are introduced according to

$$u_{i(0)}^{(\alpha)} = J(x_3, t)v_i^{(\alpha)}, \tag{30}$$

$$(\sigma_{ii(0)}^{(\alpha)} - \lambda^{(\alpha)}u_{3(0),3}^{(\alpha)}, \sigma_{12(0)}^{(\alpha)}) = J(x_3, t)(\tau_{ii}^{(\alpha)}, \tau_{12}^{(\alpha)}), \text{ no sum on } i. \tag{31}$$

In terms of the new variables the in-plane stress problem becomes

$$\tau_{ij,i}^{(\alpha)} = 0, \tag{32}$$

$$\tau_{ij}^{(\alpha)} = \lambda^{(\alpha)}v_{l,l}^{(\alpha)}\delta_{ij} + \mu^{(\alpha)}(v_{i,j}^{(\alpha)} + v_{j,i}^{(\alpha)}), \tag{33}$$

$$v_i^{(2)}\nu_i^{(2)} = 0, \epsilon_{3ij}\tau_{ki}^{(2)}\nu_k^{(2)}\nu_j^{(2)} = 0 \text{ on } \mathscr{C}, \tag{34}$$

$$v_i^{(1)} = v_i^{(2)}, (\tau_{ij}^{(2)} - \tau_{ij}^{(1)})\nu_i^{(1)} + \nu_j^{(1)} = 0 \text{ on } \mathscr{S}. \tag{35}$$

If (23) is now averaged and $u_{i(0)}^{(\alpha)}$ is substituted from (30), the constitutive equations for partial axial stresses become

$$\sigma_{33}^{(\alpha p)} = (\lambda^{(\alpha p)} + 2\mu^{(\alpha p)})u_{3,3}^{(\alpha a)} + (-1)^{\alpha+1}K^{(\alpha)}J(x_3, t) \tag{36}$$

where

$$K^{(\alpha)} = \frac{\lambda^{(\alpha)}}{A}\oint_{\mathscr{S}} v_i^{(\alpha)}\nu_i^{(1)}\,ds. \tag{37}$$

Clearly, the calculation of the constants $K^{(\alpha)}$ requires that the boundary value problem (32)–(35) be solved.

(ii) *Constitutive relations for interaction term.* It still remains to relate the interaction term P in (15) to the constituent displacements. For this purpose the axial shear problem (22)–(27) must be considered. Since $u_{3(0)}^{(\alpha)}$ is independent of the in-plane coordinates, with (23), (24) and (30), eqn (22) becomes

$$\mu^{(\alpha)}u_{3(2),ii}^{(\alpha)} = C^{(\alpha)}(x_3, t) - (\lambda^{(\alpha)} + \mu^{(\alpha)})v_{i,i}^{(\alpha)}J_{,3}(x_3, t). \tag{38}$$

The function $C^{(\alpha)}(x_3, t)$ can be related to P by integration of (38) over $A^{(\alpha)}$ using (16), (25) and the Gauss Theorem, to obtain

$$C^{(\alpha)} = -\frac{(-1)^{\alpha}}{n^{(\alpha)}}[P + K^{(\alpha)}J_{,3}]. \tag{39}$$

If (39) is now substituted in (38), $u_{3(2)}^{(\alpha)}$ is seen to satisfy the Poisson's equation,

$$\mu^{(\alpha)}u_{3(2),ii}^{(\alpha)} = -(-1)^{\alpha}\left[\frac{P}{n^{(\alpha)}} + \left\{\frac{K^{(\alpha)}}{n^{(\alpha)}} + (-1)^{\alpha}(\lambda^{(\alpha)} + \mu^{(\alpha)})v_{i,i}^{(\alpha)}\right\}J_{,3}\right] \tag{40}$$

with boundary conditions

$$u_{3(2),i}^{(2)}\nu_i^{(2)} = 0 \text{ on } \mathscr{C}, \tag{41}$$

$$u_{3(2)}^{(2)} - u_{3(2)}^{(1)} + (u_{3(0)}^{(2)} - u_{3(0)}^{(1)})/\epsilon^2 = 0 \quad \text{on } \mathscr{S}, \tag{42}$$

$$(\mu^{(2)}u_{3(2)}^{(2)} - \mu^{(1)}u_{3(2)}^{(1)})_{,i}\nu_i^{(1)} + \frac{1}{2}(\mu^{(2)} - \mu^{(1)})(v_i^{(1)} + v_i^{(2)})\nu_i^{(1)}J_{,3} = 0 \quad \text{on } \mathscr{S} \tag{43}$$

where (43) follows from (27) using (24) and (30).

Because of the Neumann type boundary condition (41), the solution of the problem (40)–(43) is unique within a function $H(x_3, t)$ independent of in-plane coordinates. Without any loss in generality this function can be included in the zeroth order displacement field by defining

$$\tilde{u}_{3(0)}^{(\alpha)} = u_{3(0)}^{(\alpha)} + \epsilon^2 H(x_3, t), \tag{44}$$

so that uniqueness of the solution of (40)–(43) can be imposed by requiring that

$$u^{(1)}_{3(2)} = 0 \quad \text{at point } 0 \in A^{(1)}. \tag{45}$$

As is evident from (40), there are two forcing functions in the axial shear problem: (a) the interaction P and (b) a term linearly related to the derivative of the jump quantity J defined by (29). Consequently, further analysis is facilitated by the introduction of variables $w^{(\alpha)}$ and $w^{*(\alpha)}$ defined by

$$u^{(1)}_{3(2)} = Pw^{*(1)} + J_{,3}w^{(1)}, \tag{46}$$

$$u^{(2)}_{3(2)} + (u^{(2)}_{3(0)} - u^{(1)}_{3(0)})/\epsilon^2 = Pw^{*(2)} + J_{,3}w^{(2)}. \tag{47}$$

Introduction of the quantity $(u^{(2)}_{3(0)} - u^{(1)}_{3(0)})$ in (47) insures that $w^{*(\alpha)}$ and $w^{(\alpha)}$ are continuous across the interface (see eqn 42). In terms of the variables $w^{*(\alpha)}$ and $w^{(\alpha)}$ the axial shear problem decomposes into two time independent problems within a cell. The variable $w^{*(\alpha)}$ satisfies

$$\mu^{(\alpha)}w^{*(\alpha)}_{;ii} = \frac{(-1)^{\alpha+1}}{n^{(\alpha)}}, \tag{48}$$

$$w^{*(2)}_{;i}\nu^{(2)}_i = 0 \text{ on } \mathscr{C}, \tag{49}$$

$$w^{*(1)} = w^{*(2)}, (\mu^{(1)}w^{*(1)}_{;i} - \mu^{(2)}w^{*(2)}_{;i})\nu^{(1)}_i = 0 \quad \text{on } \mathscr{S}, \tag{50}$$

whereas $w^{(\alpha)}$ is the solution of

$$\mu^{(\alpha)}w^{(\alpha)}_{,ii} = (-1)^{\alpha+1}\frac{K^{(\alpha)}}{n^{(\alpha)}} - (\lambda^{(\alpha)} + \mu^{(\alpha)})v^{(\alpha)}_{i,i}, \tag{51}$$

$$w^{(2)}_{,i}\nu^{(2)}_i = 0 \text{ on } \mathscr{C}, \tag{52}$$

$$w^{(1)} = w^{(2)} \text{ on } \mathscr{S}, \tag{53}$$

$$\left[\mu^{(2)}w^{(2)}_{,i} - \mu^{(1)}w^{(1)}_{,i} + \frac{1}{2}(\mu^{(2)} - \mu^{(1)})(v^{(1)}_i + v^{(2)}_i)\right]\nu^{(1)}_i = 0 \quad \text{on } \mathscr{S}. \tag{54}$$

To be added to the above is the condition (45) for uniqueness of $u^{(\alpha)}_{3(2)}$, i.e.

$$w^{*(1)} = 0,\ w^{(1)} = 0 \quad \text{at } 0 \in A^{(1)}. \tag{55}$$

It is of some interest to point out that the boundary value problem for $w^{*(\alpha)}$ is exactly the same as the microstructure boundary value problem encountered in an analogous treatment of a heat conduction problem [3].

Once the solutions to the boundary value problems for $w^{(\alpha)}$ and $w^{*(\alpha)}$ have been obtained, the axial displacement field can be determined from

$$u_3{}^{(\alpha)} = \tilde{u}^{(1)}_{3(0)} + \epsilon^2\{Pw^{*(\alpha)} + J_{,3}w^{(\alpha)}\} + 0(\epsilon^4), \tag{56}$$

which follows from (17a), (46) and (47). On averaging, eqn (56) furnishes

$$u_3{}^{(\alpha\alpha)} = \tilde{u}^{(1)}_{3(0)} + \epsilon^2\{Pw^{*(\alpha\alpha)} + J_{,3}w^{(\alpha\alpha)}\} + O(\epsilon^4). \tag{57}$$

If $\tilde{u}^{(1)}_{3(0)}$ is eliminated from the two equations implied by (57), constitutive equations for the interaction term are obtained in the form

$$P = \beta(u_3{}^{(2\alpha)} - u_3{}^{(1\alpha)})/\epsilon^2 + \gamma J_{,3} \tag{58}$$

where

$$\beta = 1/(w^{*(2a)} - w^{*(1a)}) \tag{59}$$

and

$$\gamma = (w^{(1a)} - w^{(2a)})/(w^{*(2a)} - w^{*(1a)}). \tag{60}$$

Together with (58) and the approximation of (29) by

$$J = \lambda^{(2)} u_{3,3}^{(2a)} - \lambda^{(1)} u_{3,3}^{(1a)}, \tag{61}$$

the foregoing analysis essentially completes the mixture formulation of the wave propagation problem. The main results are the expressions (37), (59) and (60) for mixture properties which are to be calculated from time independent boundary value problems given by: (i) eqns (32)–(35) for $v_i^{(\alpha)}$, (ii) eqns (48)–(50), (55) for $w^{*(\alpha)}$ and (iii) eqns (51)–(55) for $w^{(\alpha)}$, all defined over the unit cell. For future reference these problems shall be termed as the in-plane stress and axial shear microstructure boundary value problems (MBVPs).

MIXTURE QUANTITIES AND RECOVERY OF MICROSTRUCTURE

To select an appropriate mixture Young's modulus $\bar{E}_{(m)}$ and mixture density $\bar{\rho}_{(m)}$ used for scaling, the mixture equations (15) are first written in terms of the average axial displacements by utilizing the relations (36) and (58);

$$[(\lambda^{(1p)} + 2\mu^{(1p)}) - (K^{(1)} + \gamma)\lambda^{(1)}]u_{3,33}^{(1a)} + [(K^{(1)} + \gamma)\lambda^{(2)}]u_{3,33}^{(2a)} - \rho^{(1p)}\ddot{u}_3^{(1a)} = -\beta(u_3^{(2a)} - u_3^{(1a)})/\epsilon^2, \tag{62}$$

$$[(K^{(2)} + \gamma)\lambda^{(1)}]u_{3,33}^{(1a)} + [(\lambda^{(2p)} + 2\mu^{(2p)}) - (K^{(2)} + \gamma)\lambda^{(2)}]u_{3,33}^{(2a)} - \rho^{(2p)}\ddot{u}_3^{(2a)} = \beta(u_3^{(2a)} - u_3^{(1a)})/\epsilon^2. \tag{63}$$

If either of the two displacements is eliminated from (62)–(63), the other satisfies

$$[(\lambda^{(1p)} + 2\mu^{(1p)}) + (\lambda^{(2p)} + 2\mu^{(2p)}) - (K^{(1)} - K^{(2)})(\lambda^{(1)} - \lambda^{(2)})]u_{3,33}^{(\alpha a)} - (\rho^{(1p)} + \rho^{(2p)})\ddot{u}_3^{(\alpha a)} + O(\epsilon^2) = 0. \tag{64}$$

Equation (64) suggests the following definitions for mixture properties

$$\bar{E}_{(m)} = n^{(1)}(\bar{\lambda}^{(1)} + 2\bar{\mu}^{(1)}) + n^{(2)}(\bar{\lambda}^{(2)} + 2\bar{\mu}^{(2)}) - (K^{(1)} - K^{(2)})(\bar{\lambda}^{(1)} - \bar{\lambda}^{(2)}), \tag{65a}$$

$$\bar{\rho}_{(m)} = n^{(1)}\bar{\rho}^{(1)} + n^{(2)}\bar{\rho}^{(2)}. \tag{65b}$$

With this selection of mixture properties, the nondimensional constituent properties that occur in the scaled equations (6)–(10) are of order unity for achievable range of fiber volume fractions.

At this stage it is appropriate to summarize the basic procedure for the use of the mixture theory developed here. For a given composite, the MBVPs are solved to obtain the mixture properties for use in (62)–(63). These equations are then solved for a given set of initial and boundary conditions to obtain the average displacement fields $u_3^{(\alpha a)}(x_i, t)$. If the displacement microstructure is desired, $\bar{u}_{3(0)}^{(1)}(x_3, t)$ is first obtained from (57), i.e.

$$\bar{u}_{3(0)}^{(1)} = [u_3^{(1a)} + u_3^{(2a)} - \epsilon^2\{P(w^{*(1a)} + w^{*(2a)}) + J_{,3}(w^{(1a)} + w^{(2a)})\}]/2. \tag{66}$$

Finally the fields $u_3^{(\alpha)}(x_1, x_2, x_3, t)$, correct to $0(\epsilon^2)$ can be calculated from (56).

In general it is difficult to obtain analytical solutions to the MBVPs except by approximating the unit cell and the fiber by concentric circles. This axisymmetric approximation is expected to be quite accurate in case of circular fibers arranged in a hexagonal array, and has been used in a manner of studies, e.g. [7, 8, 15]. However, in case of more general geometries, the degree of accuracy of his approximation is not immediately obvious. For this reason we have developed a

variational principle based finite element procedure for the solution of the MBVPs. The details of the computational procedure are described in the second part of the paper.

CONCLUDING REMARKS

Mixture equations have been derived for propagation of longitudinal waves in fibrous composites containing cylindrical fibers. Time independent microstructure boundary value problems have been derived whose solution can be used to determine the mixture constitutive properties. Solution of the microstructure boundary value problems and application of the mixture theory to composites with several microstructural geometries of interest form the subject of the second part of this paper.

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